

# On the Rectilinear Crossing Number of Complete Uniform Hypergraphs

Anurag Anshu\*

Rahul Gangopadhyay†

Saswata Shannigrahi‡§

Satyanarayana Vusirikala¶

February 26, 2016

## Abstract

In this paper, we consider a generalized version of the rectilinear crossing number problem of drawing complete graphs on a plane. The minimum number of crossing pairs of hyperedges in the  $d$ -dimensional rectilinear drawing of a  $d$ -uniform hypergraph is known as the  $d$ -dimensional rectilinear crossing number of the hypergraph. The currently best-known lower bound on the  $d$ -dimensional rectilinear crossing number of a complete  $d$ -uniform hypergraph with  $n$  vertices in general position in  $\mathbb{R}^d$  is  $\Omega(\frac{2^d}{\sqrt{d}} \log d) \binom{n}{2d}$ . In this paper, we improve this lower bound to  $\Omega(2^d) \binom{n}{2d}$ . We also consider the special case when all the vertices of a  $d$ -uniform hypergraph are placed on the  $d$ -dimensional moment curve. For such complete  $d$ -uniform hypergraphs with  $n$  vertices, we show that the number of crossing pairs of hyperedges is  $\Theta(\frac{4^d}{\sqrt{d}}) \binom{n}{2d}$ .

**Keywords:** Geometric Hypergraph; Crossing Simplices; Ham-Sandwich Theorem; Gale Transform; Moment Curve

## 1 Introduction

Graph drawing in a plane has been a well-studied area of research for many years with applications in various fields of computer science, e.g., CAD, database design and circuit schematics [6, 12]. One particularly interesting drawing of a graph is the rectilinear drawing, defined as an embedding of the graph on a plane ( $\mathbb{R}^2$ ) with vertices placed in general position (i.e., no three vertices are collinear) and edges connecting corresponding vertices as straight line segments. The rectilinear crossing number of a graph is defined as the minimum number of crossing pairs of edges over all rectilinear drawings of the graph [2]. Let us denote the rectilinear crossing number of a complete graph with  $n$  vertices by  $\overline{cr}_2(K_n)$ . The currently best-known lower and upper bounds on  $\overline{cr}_2(K_n)$  are  $0.37997 \binom{n}{4} + \Theta(n^3)$  and  $0.380473 \binom{n}{4} + \Theta(n^3)$ , respectively [1, 4].

The concept of a hypergraph is a natural extension to the notion of a graph. A hypergraph  $H$  is defined as a pair  $H = (V, E)$ , where  $V$  is the set of vertices and  $E$  is a set of non-empty subsets of  $V$  called hyperedges. A hypergraph in which each hyperedge contains exactly  $d$  vertices is called a  $d$ -uniform hypergraph. Throughout this paper, we consider  $d \geq 2$  because the hyperedge set of

\*National University of Singapore, Singapore. Email: a0109169@u.nus.edu

†IIT Guwahati, Assam 781039, India. Email: r.gangopadhyay@iitg.ernet.in

‡IIT Guwahati, Assam 781039, India. Email: saswata.sh@iitg.ernet.in

§Corresponding Author

¶IIT Guwahati, Assam 781039, India. Email: vusirikala@iitg.ernet.in

a 1-uniform hypergraph is only a collection of sets containing one vertex each. Let us denote the complete  $d$ -uniform hypergraph with  $n$  vertices by  $K_n^d$ . A  $d$ -dimensional rectilinear drawing of a  $d$ -uniform hypergraph with  $n \geq 2d$  vertices is defined as an embedding of the hypergraph in  $\mathbb{R}^d$  with vertices placed in general position (i.e., no  $d + 1$  of the vertices lie on a common hyperplane) and hyperedges drawn as  $(d - 1)$ -simplices spanned by the  $d$  vertices in the corresponding hyperedges [2]. In a  $d$ -dimensional rectilinear drawing of a hypergraph, two hyperedges are said to be *intersecting* if they contain a common point in their relative interiors [3]. Two intersecting hyperedges are said to be *crossing* if they are vertex disjoint, as shown in Figure 1. This definition is extended to define the crossing between a  $u$ -simplex and a  $v$ -simplex such that  $0 \leq u \leq d - 1$ ,  $0 \leq v \leq d - 1$  and all the vertices (0-faces) belonging to both these simplices are in general position in  $\mathbb{R}^d$ . Note that a  $u$ -simplex (similarly,  $v$ -simplex) in  $\mathbb{R}^d$  is defined as the convex hull  $\text{Conv}(A)$  of a set  $A$  of  $u + 1$  points ( $v + 1$  points) in general position in  $\mathbb{R}^d$ . Such a  $u$ -simplex and a  $v$ -simplex are said to be crossing if they are vertex disjoint and contain a common point in their relative interiors. The  $d$ -dimensional rectilinear crossing number of a  $d$ -uniform hypergraph is defined as the minimum number of crossing pairs of hyperedges over all  $d$ -dimensional rectilinear drawings of the hypergraph. Let  $\overline{cr}_d(H)$  denote the  $d$ -dimensional rectilinear crossing number of a  $d$ -uniform hypergraph  $H$ . In this paper, we use  $c_d$  to denote  $\overline{cr}_d(K_{2d}^d)$ . It follows that  $\overline{cr}_d(K_n^d) \geq c_d \binom{n}{2d}$ , as the set of  $d$ -dimensional rectilinear crossings created by the  $\binom{2d}{d}$  hyperedges formed by a particular set of  $2d$  vertices is disjoint from the set of  $d$ -dimensional rectilinear crossings created by the  $\binom{2d}{d}$  hyperedges formed by another set of  $2d$  vertices. The best-known lower bound on  $c_d$  is  $\Omega(\frac{2^d \log d}{\sqrt{d}})$  [2]. Currently, the only known upper bound on  $c_d$  is the trivial bound  $c_d \leq \binom{2d}{d} = \Theta(\frac{4^d}{\sqrt{d}})$ . This significant gap between the currently best-known lower and upper bounds on  $c_d$  shows that at least one of these bounds can be improved. In this paper, we work towards improving the lower bound on  $c_d$ .

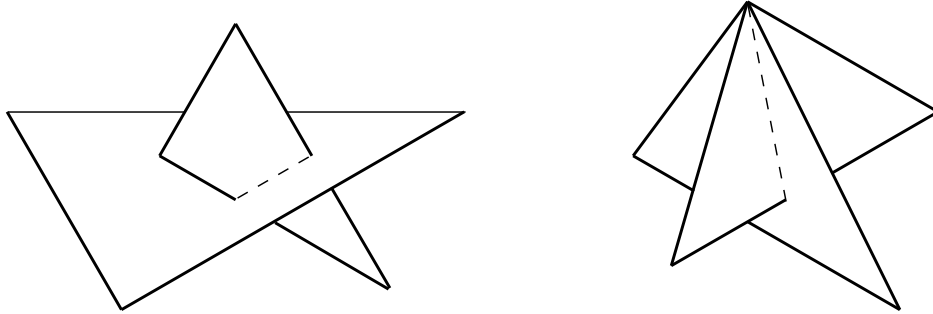


Figure 1: (left) crossing simplices in  $\mathbb{R}^3$ , (right) intersecting simplices in  $\mathbb{R}^3$

Similar to the rectilinear drawing, the convex drawing of a graph is also an active area of research [10]. The convex drawing of a graph  $G$  with  $n$  vertices is defined as an embedding of the graph  $G$  in  $\mathbb{R}^2$  with all its  $n$  vertices placed as the vertices of a convex  $n$ -gon (note that it also ensures that the vertices are in general position) and edges connecting the corresponding vertices drawn as straight line segments [11]. Let  $cr^*(G)$  denote the convex crossing number of  $G$ , defined as the minimum number of crossing pairs of edges over all convex drawings of  $G$  [7]. A  $d$ -dimensional convex drawing of a  $d$ -uniform hypergraph with  $n \geq 2d$  vertices is defined as an embedding of the hypergraph in  $\mathbb{R}^d$  with vertices placed in convex as well as general position and hyperedges drawn

as  $(d-1)$ -simplices spanned by the  $d$  vertices in the corresponding hyperedges. The  $d$ -dimensional convex crossing number of a  $d$ -uniform hypergraph is defined as the minimum number of crossing pairs of hyperedges over all  $d$ -dimensional convex drawings of the hypergraph. Let  $cr_d^*(H)$  denote the  $d$ -dimensional convex crossing number of a  $d$ -uniform hypergraph  $H$ . In this paper, we use  $c_d^*$  to denote  $cr_d^*(K_{2d}^d)$ . Since it is easy to see that  $cr_d^*(K_n^d) \geq c_d^* \binom{n}{2d}$ , finding the value of  $c_d^*$  is also an interesting problem. The above-mentioned result of Anshu and Shannigrahi [2] implies that the lower bound on  $c_d^*$  is  $\Omega(\frac{2^d \log d}{\sqrt{d}})$ . On the other hand, a trivial upper bound on  $c_d^*$  is  $O(\frac{4^d}{\sqrt{d}})$ . We are not aware of better lower and upper bounds on  $c_d^*$  in the literature. We address a special case of the problem in this paper, which is when all the vertices of  $K_{2d}^d$  lie on the  $d$ -dimensional moment curve  $\gamma = \{(t, t^2, t^3, \dots, t^d) : t \in \mathbb{R}\}$ . Our motivation to work with this special case is the Upper Bound Theorem [9] which states that the  $d$ -dimensional cyclic polytope (i.e., the polytope whose vertices are all placed on the  $d$ -dimensional moment curve) has the maximum number of faces among all  $d$ -dimensional convex polytopes having  $n$  vertices. In this paper, we explore if the placement of  $n$  vertices on the  $d$ -dimensional moment curve also maximizes the number of crossing pairs of hyperedges in a  $d$ -dimensional convex drawing of  $K_n^d$ . We use  $c_d^m$  to denote the number of crossing pairs of hyperedges of  $K_{2d}^d$ , when all the vertices of  $K_{2d}^d$  are placed as the vertices of a  $d$ -dimensional cyclic polytope.

## 1.1 Previous Works

As mentioned earlier, the best-known lower bound on  $c_d$  is  $\Omega(\frac{2^d \log d}{\sqrt{d}})$ . The proof uses Gale Transform to reduce the crossing number problem to a linear separation problem. The Gale Transform is a technique to transform a sequence of  $m$  points  $P = \langle p_1, p_2, \dots, p_m \rangle$  in  $\mathbb{R}^d$  to a sequence of  $m$  vectors  $D(P) = \langle v_1, v_2, \dots, v_m \rangle$  in  $\mathbb{R}^{m-d-1}$  for  $m \geq d+1$ . (In Section 2, we discuss the Gale Transform and its properties in detail.) A *linear separation* of a vector sequence  $D(P)$  is the partitioning of  $D(P)$  by a hyperplane, passing through the origin, into two sets. A linear separation of  $D(P)$  is called *proper* if one of the sets contains  $\lceil \frac{m}{2} \rceil$  vectors and the other contains  $\lfloor \frac{m}{2} \rfloor$  vectors. For a given set of  $d+4$  points in  $\mathbb{R}^d$ , the Gale Transform ensures that there exists a bijection between the crossing pairs of  $\lfloor \frac{d+2}{2} \rfloor$  and  $\lceil \frac{d+2}{2} \rceil$ -simplices in  $\mathbb{R}^d$  and the proper linear separations of  $d+4$  vectors in  $\mathbb{R}^3$  [8]. In order to calculate the lower bound on  $c_d$ , Anshu and Shannigrahi [2] chose a set of  $d+4$  vertices from the set of  $2d$  vertices of  $K_{2d}^d$  in  $\mathbb{R}^d$ . The Gale Transform of these  $d+4$  vertices gives  $d+4$  vectors in general position in  $\mathbb{R}^3$  (i.e., any subset containing 3 vectors spans  $\mathbb{R}^3$ ). To get a proper linear separation of this set of  $d+4$  vectors in  $\mathbb{R}^3$ , they used the Ham-Sandwich Theorem stated below.

**Ham-Sandwich Theorem.** [8] *There exists a  $(d-1)$ -hyperplane  $h$  which simultaneously bisects  $d$  finite point sets  $P_1, P_2, \dots, P_d$  in  $\mathbb{R}^d$ , such that each of the open half-spaces created by  $h$  contains at most  $\lfloor \frac{|P_i|}{2} \rfloor$  points for each of the sets  $P_i, 1 \leq i \leq d$ .*

Using this Theorem, Anshu and Shannigrahi [2] proved the existence of  $\Theta(\log d)$  distinct proper linear separations of the set of  $d+4$  vectors mentioned above. As discussed earlier, each proper linear separation of  $d+4$  vectors in  $\mathbb{R}^3$  corresponds to a crossing between  $\lfloor \frac{d+2}{2} \rfloor$  and  $\lceil \frac{d+2}{2} \rceil$  simplices in  $\mathbb{R}^d$ . They extended the crossings between the lower dimensional simplices (crossings between  $\lfloor \frac{d+2}{2} \rfloor$  and  $\lceil \frac{d+2}{2} \rceil$ -simplices) to the crossings between  $(d-1)$ -simplices to get the bound

$c_d = \Omega(\frac{2^d \log d}{\sqrt{d}})$ . In particular, they showed that  $c_4 \geq 4$ . They also constructed an arrangement of 8 vertices of a complete 4-uniform hypergraph in  $\mathbb{R}^4$  having 4 crossing pairs of simplices. This arrangement established that  $c_4 = 4$ . This is the first non-trivial result on  $c_d$  for small values of  $d$ , as it is easy to observe that  $c_2 = 0$  and  $c_3 = 1$ .

## 1.2 Our Contribution

In this paper, we first improve the lower bound on  $c_d$  in Section 3.

**Theorem 1.** *The  $d$ -dimensional rectilinear crossing number of a complete  $d$ -uniform hypergraph with  $2d$  vertices in general position in  $\mathbb{R}^d$  is  $\Omega(2^d)$ .*

**Corollary 1.**  $\overline{cr}_d(K_n^d) = \Omega(2^d) \binom{n}{2d}$ .

We derive the exact value of  $c_d^m$  in Section 4. For a sufficiently large  $d$ , it implies that there exists a constant  $c > 0$  such that the number of crossing pairs of hyperedges when all  $n$  vertices of  $K_n^d$  are placed on the  $d$ -dimensional moment curve is at least  $c$  times the number of crossing pairs of hyperedges in any  $d$ -dimensional convex drawing of  $K_n^d$ .

**Theorem 2.**

$$c_d^m = \begin{cases} \binom{2d-1}{d-1} - \sum_{i=1}^{\frac{d}{2}} \binom{d}{i} \binom{d-1}{i-1} & \text{if } d \text{ is even} \\ \binom{2d-1}{d-1} - 1 - \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-1}{i} \binom{d}{i} & \text{if } d \text{ is odd} \end{cases}$$

$$= \Theta\left(\frac{4^d}{\sqrt{d}}\right)$$

**Corollary 2.** *The number of crossing pairs of hyperedges when all the  $n$  vertices of  $K_n^d$  are placed on the  $d$ -dimensional moment curve is  $\Theta\left(\frac{4^d}{\sqrt{d}}\right) \binom{n}{2d}$ .*

In Section 5, we prove the following Theorem which implies that the 3-dimensional convex crossing number of  $K_6^3$  is 3. It also shows that the placement of  $n$  vertices on the 3-dimensional moment curve maximizes the number of crossing pairs of hyperedges in a 3-dimensional convex drawing of  $K_n^3$ . For  $d > 3$ , we do not know if the placement of  $n$  vertices on the  $d$ -dimensional moment curve maximizes the number of crossing pairs of hyperedges in a  $d$ -dimensional convex drawing of  $K_n^d$ .

**Theorem 3.** *The number of crossing pairs of hyperedges of  $K_6^3$  is 3 when all the vertices of  $K_6^3$  are in convex as well as general position in  $\mathbb{R}^3$ .*

**Corollary 3.**  $cr_3^*(K_n^3) = 3 \binom{n}{6}$ .

## 2 Gale Transform

In this Section, we discuss the Gale Transform and its properties in detail. As mentioned earlier, the Gale Transform transforms a sequence of  $m \geq d+1$  points  $P = \langle p_1, p_2, \dots, p_m \rangle$  in  $\mathbb{R}^d$  (such that the affine hull of the points  $p_1, p_2, \dots, p_m$  is  $\mathbb{R}^d$ ) to a sequence of  $m$  vectors  $D(P) = \langle v_1, v_2, \dots, v_m \rangle$

in  $\mathbb{R}^{m-d-1}$ . When  $m \leq 2d$ , this technique helps in analyzing the properties of the point sequence  $P$  by analyzing the properties of  $D(P)$  in a lower dimensional space. Let  $p_i$  denote the  $i^{th}$  point in the point sequence  $P$  with coordinates  $(x_1^i, x_2^i, \dots, x_d^i)$ . To obtain the Gale Transform of  $P$ , let us consider the following matrix  $M(P)$ .

$$M(P) = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \ddots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

The rank of the matrix  $M(P)$  is  $d+1$  since there exists a set of  $d+1$  points in  $P$  that are affinely independent. This implies that the dimension of the null space of the row space of  $M(P)$  is  $m-d-1$ . Let  $\{(b_1^1, b_2^1, \dots, b_m^1), (b_1^2, b_2^2, \dots, b_m^2), \dots, (b_1^{m-d-1}, b_2^{m-d-1}, \dots, b_m^{m-d-1})\}$  be a basis of this null space. The Gale Transform of the point sequence  $P$  is the sequence of  $m$  vectors  $D(P) = \langle (b_1^1, b_2^1, \dots, b_1^{m-d-1}), (b_2^1, b_2^2, \dots, b_2^{m-d-1}), \dots, (b_m^1, b_m^2, \dots, b_m^{m-d-1}) \rangle$ . Note that  $D(P)$  can also be considered as a point sequence in  $\mathbb{R}^{m-d-1}$  for a particular choice of the basis. Since the basis of the null space of the row space of  $M(P)$  is not unique, it implies that the Gale Transform of  $P$  is not unique. However, the following properties of the Gale Transform can be easily observed for any choice of the basis. For the sake of completeness, we give proofs for these observations.

**Lemma 1.** [8] *Every set of  $m-d-1$  vectors in  $D(P)$  spans  $\mathbb{R}^{m-d-1}$  if the points in  $P$  are in general position.*

*Proof.* Without loss of generality, let us assume that the first  $m-d-1$  vectors in  $D(P)$ , i.e.,  $(b_1^1, b_2^1, \dots, b_1^{m-d-1}), (b_2^1, b_2^2, \dots, b_2^{m-d-1}), \dots, (b_{m-d-1}^1, b_{m-d-1}^2, \dots, b_{m-d-1}^{m-d-1})$  do not span  $\mathbb{R}^{m-d-1}$ . In other words, we assume that these vectors are linearly dependent. This implies that there exist real numbers  $\lambda_1, \lambda_2, \dots, \lambda_{m-d-1}$ , not all of them zero, such that  $\lambda_1(b_1^1, b_2^1, \dots, b_1^{m-d-1}) + \lambda_2(b_2^1, b_2^2, \dots, b_2^{m-d-1}) + \dots + \lambda_{m-d-1}(b_{m-d-1}^1, b_{m-d-1}^2, \dots, b_{m-d-1}^{m-d-1}) = \vec{0}$ . Let us consider the vector  $(\lambda_1, \lambda_2, \dots, \lambda_{m-d-1}, \lambda_{m-d} = 0, \dots, \lambda_m = 0)$ . It is easy to see that  $\lambda_1(b_1^1, b_2^1, \dots, b_1^{m-d-1}) + \lambda_2(b_2^1, b_2^2, \dots, b_2^{m-d-1}) + \dots + \lambda_m(b_m^1, b_m^2, \dots, b_m^{m-d-1}) = \vec{0}$ . This implies that the vector  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  lies in the row space of  $M(P)$ . This further implies that there exist real numbers  $\alpha_1, \alpha_2, \dots, \alpha_{d+1}$ , not all of them zero, such that the following set of linear equations holds for each  $i$  satisfying  $1 \leq i \leq m$ .

$$\alpha_1 x_1^i + \alpha_2 x_2^i + \dots + \alpha_d x_d^i + \alpha_{d+1} = \lambda_i$$

This implies the last  $d+1$  points in  $P$ , i.e.,  $\{p_{m-d}, p_{m-d+1}, \dots, p_m\}$  lie on the hyperplane  $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_d x_d + \alpha_{d+1} = 0$ . This is a contradiction to the assumption that points in  $P$  are in general position in  $\mathbb{R}^d$ . This completes the proof.  $\square$

**Lemma 2.** [8] *Consider 2 integers  $u$  and  $v$  satisfying  $1 \leq u \leq d-1$ ,  $1 \leq v \leq d-1$  and  $u+v+2 = m$ . If the points of  $P$  are in general position in  $\mathbb{R}^d$ , there exists a bijection between the crossing pairs of  $u$  and  $v$ -simplices in  $P$  and linear separations of  $D(P)$  into  $D(P_1)$  and  $D(P_2)$  such that  $|D(P_1)| = u+1$  and  $|D(P_2)| = v+1$ .*

*Proof.* ( $\Rightarrow$ ) Let  $\sigma$  be a  $u$ -simplex that crosses a  $v$ -simplex  $\nu$ , such that  $1 \leq u \leq d-1$ ,  $1 \leq v \leq d-1$  and  $u+v+2 = m$ . Without loss of generality, we assume that  $\sigma$  is spanned by the first  $u+1$  points  $\{p_1, p_2, \dots, p_{u+1}\}$  of  $P$  and  $\nu$  is spanned by the remaining  $v+1$  points  $\{p_{u+2}, p_{u+3}, \dots, p_m\}$  of  $P$ . As there exists a crossing between  $\sigma$  and  $\nu$ , we know that there exists a point  $p$  belonging to the

relative interiors of both  $\sigma$  and  $\nu$ . This implies that there exist real numbers  $\lambda_k \geq 0$ ,  $1 \leq k \leq m$ , satisfying the following equations:

$$\begin{aligned} p &= \sum_{i \in \{1, 2, \dots, u+1\}} \lambda_i p_i = \sum_{j \in \{u+2, u+3, \dots, m\}} \lambda_j p_j \\ \sum_{i \in \{1, 2, \dots, u+1\}} \lambda_i &= \sum_{j \in \{u+2, u+3, \dots, m\}} \lambda_j = 1 \end{aligned}$$

Therefore, we obtain the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{u+1} \\ -\lambda_{u+2} \\ \vdots \\ -\lambda_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1)$$

It is evident from Equation 1 that the vector  $(\lambda_1, \lambda_2, \dots, \lambda_{u+1}, -\lambda_{u+2}, \dots, -\lambda_m)$  lies in the null space of the row space of  $M(P)$ . This implies that  $(\lambda_1, \lambda_2, \dots, \lambda_{u+1}, -\lambda_{u+2}, \dots, -\lambda_m) = \alpha_1(b_1^1, b_2^1, \dots, b_m^1) + \alpha_2(b_1^2, b_2^2, \dots, b_m^2) + \dots + \alpha_{m-d-1}(b_1^{m-d-1}, b_2^{m-d-1}, \dots, b_m^{m-d-1})$ , for some real numbers  $\alpha_1, \alpha_2, \dots, \alpha_{m-d-1}$ , not all of them zero. In other words, there exist  $\alpha_1, \alpha_2, \dots, \alpha_{m-d-1}$ , not all of them zero, such that  $\alpha_1 b_i^1 + \alpha_2 b_i^2 + \dots + \alpha_{m-d-1} b_i^{m-d-1} > 0$  for  $i = 1, 2, \dots, u+1$ , and  $\alpha_1 b_j^1 + \alpha_2 b_j^2 + \dots + \alpha_{m-d-1} b_j^{m-d-1} < 0$  for  $j = u+2, u+3, \dots, m$ . This shows that the hyperplane  $\sum_{i \in \{1, 2, \dots, m-d-1\}} \alpha_i x_i = 0$  separates the first  $u+1$  vectors in  $D(P)$  from the remaining  $v+1$  vectors.

( $\Leftarrow$ ) Without loss of generality, let us assume that the hyperplane

$$\sum_{i \in \{1, \dots, m-d-1\}} \alpha'_i x_i = 0$$

separates the first  $u+1$  vectors in  $D(P)$  from the remaining  $v+1$  vectors. This implies that there exists a vector  $(\mu'_1, \mu'_2, \dots, \mu'_m) = \alpha'_1(b_1^1, b_2^1, \dots, b_m^1) + \alpha'_2(b_1^2, b_2^2, \dots, b_m^2) + \dots + \alpha'_{m-d-1}(b_1^{m-d-1}, b_2^{m-d-1}, \dots, b_m^{m-d-1})$  such that the signs of  $\mu'_i$  for  $1 \leq i \leq u+1$  are opposite to the signs of  $\mu'_j$  for  $u+2 \leq j \leq m$ . Without loss of generality, let us assume that  $\mu'_i > 0$  for  $1 \leq i \leq u+1$  and  $\mu'_j < 0$  for  $u+2 \leq j \leq m$ . As this vector  $(\mu'_1, \mu'_2, \dots, \mu'_m)$  lies in the null space of the row space of  $M(P)$ , it satisfies the following equation.

$$\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^m \\ x_2^1 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & \vdots \\ x_d^1 & x_d^2 & \cdots & x_d^m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu'_1 \\ \mu'_2 \\ \vdots \\ \mu'_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (2)$$

From Equation 2, we obtain the following.

$$\sum_{i \in \{1, 2, \dots, u+1\}} \mu'_i p_i = \sum_{j \in \{u+2, u+3, \dots, m\}} -\mu'_j p_j$$

$$\sum_{i \in \{1, 2, \dots, u+1\}} \mu'_i = \sum_{j \in \{u+2, u+3, \dots, m\}} -\mu'_j$$

Rearranging the above equations, we obtain the following.

$$\begin{aligned} \sum_{i \in \{1, 2, \dots, u+1\}} \frac{\mu'_i}{\sum_{k \in \{1, 2, \dots, u+1\}} \mu'_k} p_i &= \sum_{j \in \{u+2, u+3, \dots, m\}} \frac{\mu'_j}{\sum_{k \in \{u+2, u+3, \dots, m\}} \mu'_k} p_j \\ \sum_{i \in \{1, 2, \dots, u+1\}} \frac{\mu'_i}{\sum_{k \in \{1, 2, \dots, u+1\}} \mu'_k} &= \sum_{j \in \{u+2, u+3, \dots, m\}} \frac{\mu'_j}{\sum_{k \in \{u+2, u+3, \dots, m\}} \mu'_k} = 1 \end{aligned}$$

It shows that there exists a crossing between the  $u$ -simplex spanned by the first  $u + 1$  points of  $P$  and the  $v$ -simplex spanned by the remaining  $v + 1$  points of  $P$ .  $\square$

An argument similar to the one used above can be used to prove the following Lemma.

**Lemma 3.** [8] *The points in  $P$  are in convex position in  $\mathbb{R}^d$  if and only if there is no linear hyperplane  $h$  with exactly one vector from  $D(P)$  on one side of  $h$ .*

In the following, we derive the Gale Transform of a sequence of  $m \geq d + 1$  points placed on the  $d$ -dimensional moment curve. Let this point sequence be  $A = \langle (t_1, (t_1)^2, \dots, (t_1)^d), (t_2, (t_2)^2, \dots, (t_2)^d), \dots, (t_m, (t_m)^2, \dots, (t_m)^d) \rangle$ , where each  $t_i$  for  $1 \leq i \leq m$  is a real number satisfying  $t_i < t_j$  for  $i < j$ . We obtain the following two Lemmas for  $m = d + 3$  and  $m = 2d$ , respectively.

**Lemma 4.** *The following sequence of 2-dimensional vectors  $D(A) = \langle v_1, v_2, \dots, v_{d+3} \rangle$  can be obtained by the Gale Transform of  $A = \langle (t_1, (t_1)^2, \dots, (t_1)^d), (t_2, (t_2)^2, \dots, (t_2)^d), \dots, (t_{d+3}, (t_{d+3})^2, \dots, (t_{d+3})^d) \rangle$ .*

$$v_i = \begin{cases} \left( (-1)^{d+1} \frac{\prod_{j \in \{1, 2, \dots, d+1\} \setminus \{i\}} (t_{d+2} - t_j)}{\prod_{k \in \{1, 2, \dots, d+1\} \setminus \{i\}} (t_k - t_i)}, (-1)^{d+1} \frac{\prod_{j \in \{1, 2, \dots, d+1\} \setminus \{i\}} (t_{d+3} - t_j)}{\prod_{k \in \{1, 2, \dots, d+1\} \setminus \{i\}} (t_k - t_i)} \right) & \text{if } i \in \{1, 2, \dots, d+1\} \\ (1, 0) & \text{if } i = d+2 \\ (0, 1) & \text{if } i = d+3 \end{cases}$$

*Proof.* Let us consider the following matrix  $M(A)$ .

$$M(A) = \begin{bmatrix} t_1 & t_2 & \cdots & t_{d+3} \\ (t_1)^2 & (t_2)^2 & \cdots & (t_{d+3})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (t_1)^d & (t_2)^d & \cdots & (t_{d+3})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

To obtain the basis of the null space, we need to find solutions of the following  $d + 1$  equations involving  $d + 3$  variables  $\gamma_1, \gamma_2, \dots, \gamma_{d+3}$ .

$$\begin{bmatrix} t_1 & t_2 & \cdots & t_{d+3} \\ (t_1)^2 & (t_2)^2 & \cdots & (t_{d+3})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (t_1)^d & (t_2)^d & \cdots & (t_{d+3})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{d+3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3)$$

Rearranging Equation 3, we get the following:

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{d+1} \end{bmatrix} = - \begin{bmatrix} t_1 & t_2 & \cdots & t_{d+1} \\ (t_1)^2 & (t_2)^2 & \cdots & (t_{d+1})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (t_1)^d & (t_2)^d & \cdots & (t_{d+1})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} t_{d+2} & t_{d+3} \\ (t_{d+2})^2 & (t_{d+3})^2 \\ \vdots & \vdots \\ (t_{d+2})^d & (t_{d+3})^d \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{d+2} \\ \gamma_{d+3} \end{bmatrix}$$

Setting  $\gamma_{d+2} = 1$  and  $\gamma_{d+3} = 0$ , we obtain the following for every  $i$  satisfying  $1 \leq i \leq d+1$ .

$$\gamma_i = (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_{d+2} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_k - t_i)}.$$

Setting  $\gamma_{d+2} = 0$  and  $\gamma_{d+3} = 1$ , we obtain the following for every  $i$  satisfying  $1 \leq i \leq d+1$ .

$$\gamma_i = (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_{d+3} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_k - t_i)}.$$

Note that the vectors  $\left( (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{1\}} (t_{d+2} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{1\}} (t_k - t_1)}, \dots, (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{d+1\}} (t_{d+2} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{d+1\}} (t_k - t_{d+1})}, 1, 0 \right)$  and  $\left( (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{1\}} (t_{d+3} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{1\}} (t_k - t_1)}, \dots, (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{d+1\}} (t_{d+3} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{d+1\}} (t_k - t_{d+1})}, 0, 1 \right)$  are linearly independent and form a basis of the null space of the row space of  $M(A)$ . Hence, the result follows.  $\square$

**Lemma 5.** *The following sequence of  $(d-1)$ -dimensional vectors  $D(A) = \langle v_1, v_2, \dots, v_{2d} \rangle$  can be obtained by the Gale Transform of  $A = \langle (t_1, (t_1)^2, \dots, (t_1)^d), (t_2, (t_2)^2, \dots, (t_2)^d), \dots, (t_{2d}, (t_{2d})^2, \dots, (t_{2d})^d) \rangle$ .*

$$v_i = \begin{cases} \left( (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_{d+2} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_k - t_i)}, (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_{d+3} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_k - t_i)}, \dots, (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_{2d} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_k - t_i)} \right) & \text{if } i \in \{1, 2, \dots, d+1\} \\ (1, 0, \dots, 0) & \text{if } i = d+2 \\ (0, 1, \dots, 0) & \text{if } i = d+3 \\ \vdots & \vdots \\ (0, 0, \dots, 1) & \text{if } i = 2d \end{cases}$$

*Proof.* Let us consider the matrix  $M(A)$ .

$$M(A) = \begin{bmatrix} t_1 & t_2 & \cdots & t_{2d} \\ (t_1)^2 & (t_2)^2 & \cdots & (t_{2d})^2 \\ \vdots & \vdots & \vdots & \vdots \\ (t_1)^d & (t_2)^d & \cdots & (t_{2d})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

To compute the basis of this null space, we solve the following equation.



$$\begin{bmatrix} t_1 & t_2 & \cdots & t_{2d} \\ (t_1)^2 & (t_2)^2 & \cdots & (t_{2d})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (t_1)^d & (t_2)^d & \cdots & (t_{2d})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{2d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (4)$$

Rearranging Equation 4, we get the following:

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{d+1} \end{bmatrix} = - \begin{bmatrix} t_1 & t_2 & \cdots & t_{d+1} \\ (t_1)^2 & (t_2)^2 & \cdots & (t_{d+1})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (t_1)^d & (t_2)^d & \cdots & (t_{d+1})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} t_{d+2} & t_{d+3} & \cdots & t_{2d} \\ (t_{d+2})^2 & (t_{d+3})^2 & \cdots & (t_{2d})^2 \\ \vdots & \vdots & \ddots & \vdots \\ (t_{d+2})^d & (t_{d+3})^d & \cdots & (t_{2d})^d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \gamma_{d+2} \\ \gamma_{d+3} \\ \vdots \\ \gamma_{2d} \end{bmatrix}$$

Consider an  $r$  satisfying  $1 \leq r \leq d-1$ . For this  $r$ , we obtain the values of  $\gamma_1, \gamma_2, \dots, \gamma_{d+1}$  by setting  $(\gamma_{d+2}, \gamma_{d+3}, \dots, \gamma_{2d}) = e_r$ , where  $e_r$  is the  $r^{\text{th}}$  row of the identity matrix  $I_{d-1}$ .

$$\gamma_i = (-1)^{d+1} \frac{\prod_{j \in \{1, 2, \dots, d+1\} \setminus \{i\}} (t_{d+1+r} - t_j)}{\prod_{k \in \{1, 2, \dots, d+1\} \setminus \{i\}} (t_k - t_i)}$$

For  $d-1$  different values of  $r$ , the  $d-1$  vectors obtained in the above-mentioned way form a basis of the null space of the row space of  $M(A)$ . Hence, the result follows.  $\square$

### 3 Lower Bound on the $d$ -Dimensional Rectilinear Crossing Number of $K_{2d}^d$

In this Section, we improve the lower bound on  $c_d$  by using the Proper Separation Theorem and Lemma 6 mentioned below. Recall that  $c_d$  denotes the  $d$ -dimensional rectilinear crossing number of  $K_{2d}^d$ . Let  $P$  denote the set of  $2d$  vertices of  $K_{2d}^d$  that are in general position in  $\mathbb{R}^d$ . Two nonempty convex sets  $C$  and  $D$  in  $\mathbb{R}^d$  are said to be *properly separated* if there exists a  $(d-1)$ -dimensional hyperplane  $h$  such that  $C$  and  $D$  lie in the opposite closed half-spaces determined by  $h$ , and  $C$  and  $D$  are not both contained in the hyperplane  $h$  [5].

**Proper Separation Theorem.** [5] *Two nonempty convex sets  $C$  and  $D$  in  $\mathbb{R}^d$  can be properly separated if and only if their relative interiors are disjoint.*

**Lemma 6.** *Consider two disjoint point sets  $U, V \subset P$  such that  $|U| = p$ ,  $|V| = q$ ,  $2 \leq p, q \leq d$  and  $p+q \geq d+1$ . If the  $(p-1)$ -simplex formed by  $U$  crosses the  $(q-1)$ -simplex formed by  $V$ , then the  $(d-1)$ -simplices formed by any two disjoint point sets  $U' \supseteq U$  and  $V' \supseteq V$  satisfying  $|U'| = |V'| = d$  also form a crossing pair.*

*Proof.* For the given disjoint sets  $U = \{x_1, x_2, \dots, x_p\}$  and  $V = \{y_1, y_2, \dots, y_q\}$ , assume for the sake of contradiction that there exist two  $(d-1)$ -simplices, formed respectively by the disjoint point sets  $U' \supseteq U$  and  $V' \supseteq V$ , that do not cross. If  $\text{Conv}(U') \cap \text{Conv}(V') = \phi$ , then it is easy to observe that  $\text{Conv}(U) \cap \text{Conv}(V) = \phi$ . This leads to a contradiction. Otherwise, consider the convex sets  $\text{Conv}(U')$  and  $\text{Conv}(V')$ . Since  $\text{Conv}(U')$  and  $\text{Conv}(V')$  do not cross, their relative interiors are disjoint. The Proper Separation Theorem guarantees that there exists a  $(d-1)$ -dimensional

hyperplane  $h$  such that  $\text{Conv}(U')$  and  $\text{Conv}(V')$  lie in the opposite closed half-spaces determined by  $h$ , which further implies that  $\text{Conv}(U)$  and  $\text{Conv}(V)$  lie in the opposite closed half-spaces determined by  $h$ . As  $\text{Conv}(U)$  and  $\text{Conv}(V)$  have a common point in their relative interiors,  $U$  and  $V$  cannot be properly separated by  $h$ . It implies that all the  $p + q$  points of  $U \cup V$  lie on the  $(d - 1)$ -dimensional hyperplane  $h$ . Since the  $p + q \geq d + 1$  points in  $U \cup V$  are in general position, this leads to a contradiction.  $\square$

**Proof of Theorem 1:** Consider the hypergraph  $K_{2d}^d$  whose vertices are in general position in  $\mathbb{R}^d$ , and let  $A$  be any subset of  $d + 3$  vertices selected from these vertices. The Gale Transform  $D(A)$  of the point set  $A$  contains  $d + 3$  vectors in  $\mathbb{R}^2$ , which can also be considered as a sequence of  $d + 3$  points (as mentioned in Section 2). In order to apply the Ham-Sandwich Theorem (mentioned in the Introduction) in  $\mathbb{R}^2$ , we assign the points in  $D(A)$  to  $P_1$  and the origin to  $P_2$  to obtain a line  $l$  passing through the origin that bisects the points in  $D(A)$  such that each partition (open half-space) contains at most  $\lfloor \frac{1}{2}|D(A)| \rfloor$  points from  $D(A)$ . Since the points in  $A$  are in general position, every pair of vectors in  $D(A)$  spans  $\mathbb{R}^2$ . Hence, at most one point from  $D(A)$  can lie on  $l$ . As a consequence,  $l$  can be rotated using the origin as the axis of rotation to obtain a proper linear separation of  $D(A)$  into 2 subsets  $l_1^+$  and  $l_1^-$  of size  $\lfloor \frac{d+3}{2} \rfloor$  and  $\lceil \frac{d+3}{2} \rceil$ , respectively, such that  $l_1^+$  denotes the positive (counter-clockwise) side and  $l_1^-$  denotes the negative (clockwise) side of  $l$ . Lemma 2 implies that this proper linear separation corresponds to a crossing pair of a  $(\lfloor \frac{d+3}{2} \rfloor - 1)$ -simplex and a  $(\lceil \frac{d+3}{2} \rceil - 1)$ -simplex in  $\mathbb{R}^d$ . We observe from Lemma 6 that this crossing pair of simplices can be used to obtain  $\binom{d-3}{\lfloor \frac{d-3}{2} \rfloor}$  distinct crossing pairs of  $(d - 1)$ -simplices formed by the vertices of the hypergraph  $K_{2d}^d$ .

We rotate  $l$  clockwise using the origin as the axis of rotation, until one of the  $d + 3$  points in  $D(A)$  moves from one side of the line  $l$  to the other side. Since every pair of vectors in  $D(A)$  spans  $\mathbb{R}^2$ , it can be observed that exactly one point of  $D(A)$  can change its side at any particular time during the rotation of  $l$ . We further rotate  $l$  clockwise to obtain another new partition  $\{l_2^+, l_2^-\}$ , each having at least  $\lfloor \frac{d+1}{2} \rfloor$  points, at the instance a point in either  $l_1^+$  or  $l_1^-$  changes its side. This new linear separation corresponds to a crossing pair of simplices in  $\mathbb{R}^d$ , which can be used to obtain at least  $\binom{d-3}{\lfloor \frac{d-5}{2} \rfloor}$  distinct crossing pairs of  $(d - 1)$ -simplices formed by the vertices of the hypergraph  $K_{2d}^d$ . Note that all the crossing pairs of simplices obtained by extending the partitions  $\{l_1^+, l_1^-\}$  and  $\{l_2^+, l_2^-\}$  are distinct. Continuing in this manner for any  $1 \leq k \leq \lfloor \frac{d-3}{2} \rfloor - 1$ , we rotate  $l$  clockwise to obtain a new partition  $\{l_{k+1}^+, l_{k+1}^-\}$ , each having at least  $\lfloor \frac{d-2k+3}{2} \rfloor$  points, at any time a point in either  $l_k^+$  or  $l_k^-$  changes its side. Therefore, the corresponding crossing pair of simplices in  $\mathbb{R}^d$  can be extended to crossing pairs of  $(d - 1)$ -simplices in at least  $\binom{d-3}{d - \lfloor \frac{d-2k+3}{2} \rfloor} = \binom{d-3}{\lfloor \frac{d-2k-3}{2} \rfloor}$  distinct ways. Hence, the number of crossing pairs of  $(d - 1)$ -simplices obtained using this method is at least

$$\binom{d-3}{\lfloor \frac{d-3}{2} \rfloor} + \binom{d-3}{\lfloor \frac{d-5}{2} \rfloor} + \binom{d-3}{\lfloor \frac{d-7}{2} \rfloor} + \dots + \binom{d-3}{1} = \Theta(2^d).$$

## 4 Number of Crossing Pairs of Hyperedges of $K_{2d}^d$ on the Moment Curve

In this Section, we obtain the value of  $c_d^m$ . Recall that  $c_d^m$  denotes the number of crossing pairs of hyperedges of  $K_{2d}^d$ , when all the  $2d$  vertices of  $K_{2d}^d$  are placed on the  $d$ -dimensional moment curve. We first prove a lower bound on  $c_d^m$  using an approach similar to the proof of Theorem 1 and show later that this bound can be improved by using other techniques to obtain the exact value of  $c_d^m$ . Let  $A$  be a subset of any  $d + 3$  vertices selected from these  $2d$  vertices. To establish a lower bound

on  $c_d^m$ , we count the number of linear separations of the vectors in  $D(A)$ , i.e., the Gale Transform of  $A$ , obtained in Lemma 4. For the given  $A = \langle (t_1, (t_1)^2, \dots, (t_1)^d), (t_2, (t_2)^2, \dots, (t_2)^d), \dots, (t_{d+3}, (t_{d+3})^2, \dots, (t_{d+3})^d) \rangle$ , where  $t_1 < t_2 < \dots < t_{d+3}$ , the  $i^{\text{th}}$  vector  $v_i$  in  $D(A)$  is the following:

$$v_i = \begin{cases} \left( (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_{d+2} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_k - t_i)}, (-1)^{d+1} \frac{\prod_{j \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_{d+3} - t_j)}{\prod_{k \in \{1,2,\dots,d+1\} \setminus \{i\}} (t_k - t_i)} \right) & \text{if } i \in \{1, 2, \dots, d+1\} \\ (1, 0) & \text{if } i = d+2 \\ (0, 1) & \text{if } i = d+3 \end{cases}$$

Note that for every  $1 \leq i \leq d+3$ , each vector  $v_i$  in  $D(A)$  is represented as an ordered pair  $(a_i, b_i)$  where  $a_i, b_i \in \mathbb{R}$ . We denote the slope of the vector  $v_i$  as  $s_i = \frac{b_i}{a_i}$ . In order to count the number of linear separations, we observe the following properties of these vectors.

**Observation 1.** *The sequence of 2-dimensional vectors  $D(A) = \langle v_1, v_2, \dots, v_{d+3} \rangle$  having slopes  $\langle s_1, s_2, \dots, s_{d+3} \rangle$  satisfies the following properties.*

- (i) *For any  $1 \leq i \leq d+1$ ,  $v_i$  lies in the first (third) quadrant if  $d+1+i$  is odd (even).*
- (ii)  $\infty = s_{d+3} > s_1 > s_2 > \dots > s_{d+1} > s_{d+2} = 0$ .

**Lemma 7.**  $c_d^m = \Omega(2^d \sqrt{d})$ .

*Proof.* Consider the vectors in  $D(A)$ , that can also be considered as a sequence of  $d+3$  points in  $\mathbb{R}^2$ . We apply the Ham-Sandwich Theorem by assigning the points in  $D(A)$  to  $P_1$  and the origin to  $P_2$  to obtain a line  $l$  passing through the origin that bisects the points in  $D(A)$  into two partitions, each containing at most  $\lfloor \frac{1}{2}|D(A)| \rfloor$  points. Since at most one point from  $D(A)$  can lie on  $l$ , it can be rotated using the origin as the axis of rotation to obtain a proper linear separation of  $D(A)$  into 2 subsets  $l_1^+$  (positive or counter-clockwise side) and  $l_1^-$  (negative or clockwise side) of size  $\lfloor \frac{d+3}{2} \rfloor$  and  $\lceil \frac{d+3}{2} \rceil$ , respectively. This proper linear separation corresponds to a crossing pair of a  $(\lfloor \frac{d+3}{2} \rfloor - 1)$ -simplex and a  $(\lceil \frac{d+3}{2} \rceil - 1)$ -simplex in  $\mathbb{R}^d$ , as shown in Lemma 2. It follows from Lemma 6 that this crossing pair of simplices can be used to obtain  $\binom{d-3}{\lfloor \frac{d-3}{2} \rfloor}$  distinct crossing pairs of  $(d-1)$ -simplices formed by the vertices of the hypergraph  $K_{2d}^d$ . We rotate  $l$  clockwise using the origin as the axis of rotation until one of the  $d+3$  points in  $D(A)$  moves from one side of the line to the other side to obtain new subsets  $\{l_2^+, l_2^-\}$ , each having at least  $\lfloor \frac{d+1}{2} \rfloor$  points. This new linear separation  $\{l_2^+, l_2^-\}$  corresponds to a crossing pair of simplices in  $\mathbb{R}^d$ , which can be used to obtain at least  $\binom{d-3}{\lfloor \frac{d-3}{2} \rfloor}$  distinct crossing pairs of  $(d-1)$ -simplices formed by the vertices of the hypergraph  $K_{2d}^d$ . Note that all the crossing pairs of simplices obtained by extending the partitions  $\{l_1^+, l_1^-\}$  and  $\{l_2^+, l_2^-\}$  are distinct. Since all the  $d+3$  points of  $A$  lie on the  $d$ -dimensional moment curve, Observation 1 implies that the sequence of vectors in  $D(A)$ , excluding  $v_{d+2}$  and  $v_{d+3}$ , lie alternatively in the first and third quadrants with increasing slopes. As a consequence, another clockwise rotation of  $l$  results in a point in  $D(A)$  changing its side at some point of time from a side having more than or equal to  $\lceil \frac{d+3}{2} \rceil$  points to the other side. This creates a new partition  $\{l_3^+, l_3^-\}$ , each containing at least  $\lfloor \frac{d+3}{2} \rfloor$  points. We continue rotating  $l$  clockwise until we obtain the partition  $\{l_1^+, l_1^-\}$  again. In this way, we obtain at least  $2\lfloor \frac{d+3}{2} \rfloor$  distinct partitions of  $D(A)$  such that each subset in a partition contains at least  $\lfloor \frac{d+1}{2} \rfloor$  points. Hence, the number of crossing pairs of hyperedges spanned by the vertices of  $K_{2d}^d$  placed on the  $d$ -dimensional moment curve is at least

$$2\lfloor \frac{d+3}{2} \rfloor \binom{d-3}{\lfloor \frac{d-3}{2} \rfloor} = \Theta(2^d \sqrt{d}).$$

□

However, we show below that this lower bound is far from being optimal. In fact, we use Lemma 8 and Lemma 9 to prove Theorem 2 that implies  $c_d^m = \Theta(\frac{4^d}{\sqrt{d}})$ . Let us define the ordering between two points  $p = \{t, (t)^2, \dots, (t)^d\}$  and  $p' = \{t', (t')^2, \dots, (t')^d\}$  on the  $d$ -dimensional moment curve by  $p \prec p'$  ( $p$  precedes  $p'$ ) if  $t < t'$ .

**Lemma 8.** [3] *Let  $p_1 \prec p_2 \prec \dots \prec p_{\lfloor \frac{d}{2} \rfloor + 1}$  and  $q_1 \prec q_2 \prec \dots \prec q_{\lceil \frac{d}{2} \rceil + 1}$  be two distinct point sequences on the  $d$ -dimensional moment curve such that  $p_i \neq q_j$  for any  $1 \leq i \leq \lfloor \frac{d}{2} \rfloor + 1$  and  $1 \leq j \leq \lceil \frac{d}{2} \rceil + 1$ . The  $\lfloor \frac{d}{2} \rfloor$ -simplex and the  $\lceil \frac{d}{2} \rceil$ -simplex, formed respectively by these point sequences, cross if and only if every interval  $(q_j, q_{j+1})$  contains exactly one  $p_i$  and every interval  $(p_i, p_{i+1})$  contains exactly one  $q_j$ .*

**Lemma 9.** [3] *Let  $P$  and  $Q$  be two vertex-disjoint  $(d-1)$ -simplices such that each of the  $2d$  vertices belonging to these simplices lies on the  $d$ -dimensional moment curve. If  $P$  and  $Q$  cross, then there exist a  $\lfloor \frac{d}{2} \rfloor$ -simplex  $U \subsetneq P$  and another  $\lceil \frac{d}{2} \rceil$ -simplex  $V \subsetneq Q$  such that  $U$  and  $V$  cross.*

**Proof of Theorem 2:** Let  $\{C, D\}$  be a pair of disjoint vertex sets, each having  $d$  vertices of  $K_{2d}^d$  placed on the  $d$ -dimensional moment curve  $\gamma = \{t, t^2, t^3, \dots, t^d : t \in \mathbb{R}\}$ . Without loss of generality, let us assume that  $C$  contains the first vertex (i.e., the vertex corresponding to the minimum value of  $t$ ) of  $K_{2d}^d$ . Note that the number of such unordered pairs  $\{C, D\}$  is  $\frac{1}{2} \binom{2d}{d} = \binom{2d-1}{d-1}$ . Let us color the vertices in  $C$  and  $D$  by red and blue, respectively, to obtain  $d$  partitions created by the red vertices. In particular, the first  $d-1$  of these partitions are between two adjacent red vertices, and the last one is after the last red vertex. It implies from Lemma 6 and Lemma 8 that the pair of  $(d-1)$ -simplices formed by the vertices in  $C$  and  $D$  cross if there exists a sequence of  $d+2$  vertices with alternating colors. Similarly, we obtain from Lemma 8 and Lemma 9 that the pair of  $(d-1)$ -simplices formed by the vertices in  $C$  and  $D$  do not cross if there does not exist any sequence of  $d+2$  vertices with alternating colors.

When  $d$  is even, the number of disjoint vertex sets  $\{C, D\}$  that do not contain any subsequence of length  $d+2$  having alternating colors is equal to the number of ways  $d$  blue vertices can be distributed among  $d$  partitions such that at most  $\frac{d}{2}$  of the partitions are non-empty. This number is equal to  $\sum_{i=1}^{\frac{d}{2}} \binom{d}{i} \binom{d-1}{i-1}$ . When  $d$  is odd, the number of disjoint vertex sets  $\{C, D\}$  that do not contain any subsequence of length  $d+2$  having alternating colors is equal to the number of ways  $d$  blue vertices can be distributed among  $d$  partitions such that at most  $\lfloor \frac{d}{2} \rfloor$  of the first  $d-1$  partitions are non-empty. This number is equal to  $\sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-1}{i} \left( \binom{d-1}{i-1} + \binom{d-1}{i} \right) + 1$ . Hence, the total number of crossing pairs of  $(d-1)$ -simplices spanned by the  $2d$  vertices placed on the  $d$ -dimensional moment curve is

$$c_d^m = \begin{cases} \binom{2d-1}{d-1} - \sum_{i=1}^{\frac{d}{2}} \binom{d}{i} \binom{d-1}{i-1} & \text{if } d \text{ is even.} \\ \binom{2d-1}{d-1} - 1 - \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-1}{i} \binom{d}{i} & \text{if } d \text{ is odd.} \end{cases}$$

## 5 Convex Crossing Number in Lower Dimensional Space

In this Section, we show that the number of crossing pairs of hyperedges of  $K_6^3$  is 3 when all the vertices of  $K_6^3$  are in convex as well as general position in  $\mathbb{R}^3$ . Note that it implies that  $c_3^* = 3$ ,

where  $c_3^*$  denotes the 3-dimensional convex crossing number of  $K_6^3$ . However, we are not aware of the exact values of  $c_d^*$  for  $d > 3$ .

**Proof of Theorem 3:** Let  $A$  be the set of vertices of  $K_6^3$  that are in convex as well as general position in  $\mathbb{R}^3$ . Let  $D(A)$  denote the Gale Transform of  $A$ . Since the points in  $A$  are in general position, Lemma 1 shows that the 6 vectors in  $D(A)$  are in general position in  $\mathbb{R}^2$ . Since the points in  $A$  are also in convex position, Lemma 3 implies that these vectors can be partitioned by a line  $l$  passing through the origin in two possible ways, i.e., the number of vectors in the opposite open half-spaces created by  $l$  can be either 4 and 2, or 3 and 3. Note that the second case is also known as a proper linear separation that corresponds to a crossing pair of 2-simplices spanned by the points in  $A$ . Without loss of generality, let us assume that  $l$  partitions the vectors in  $D(A)$  in such a way that one of the open half-spaces created by  $l$  contains 4 vectors and the other contains 2 vectors. We rotate  $l$  clockwise using the origin as the axis of rotation until one vector changes its side. Since Lemma 3 shows that  $l$  cannot partition the vectors such that there exists 1 vector on one of its side, this new partition obtained by rotating  $l$  is a proper linear separation. We again rotate  $l$  clockwise using the origin as the axis of rotation until one vector changes its side to obtain a new partition having 4 vectors on one side and 2 on the other side. We continue rotating  $l$  clockwise till we reach the first partition to obtain three proper linear separations of the vectors in  $D(A)$ .  $\square$

## 6 Concluding Remarks

In this paper, we have improved the lower bound on the  $d$ -dimensional rectilinear crossing number of  $K_{2d}^d$ . However, there is still a significant gap between the best-known asymptotic lower and upper bounds on this number. Moreover, Anshu and Shannigrahi [2] mentioned that the exact values of the  $d$ -dimensional rectilinear crossing number of  $K_{2d}^d$  are not known for  $d > 4$ . Similarly, we mentioned in this paper that the exact values of the  $d$ -dimensional convex crossing number of  $K_{2d}^d$  are not known for  $d > 3$ . For  $d > 3$ , it is also an exciting open problem to prove or disprove the following conjecture.

**Conjecture 1.** *The placement of  $n$  vertices on the  $d$ -dimensional moment curve maximizes the number of crossing pairs of hyperedges in a  $d$ -dimensional convex drawing of  $K_n^d$ .*

## Acknowledgement

This work has been supported by Ramanujan Fellowship, Department of Science and Technology, Government of India, grant number SR/S2/RJN-87/2011.

## References

- [1] B. M. Ábrego, M. Cetina, S. Fernández-Merchant, J. Leños and G. Salazar. On  $(\leq k)$ -edges, crossings, and halving lines of geometric drawings of  $K_n$ . *Discrete and Computational Geometry*, 48, 192-215 (2012).
- [2] A. Anshu and S. Shannigrahi. A lower bound on the crossing number of uniform hypergraphs. *Discrete Applied Mathematics*, <http://dx.doi.org/10.1016/j.dam.2015.10.009> (2015) (in press).
- [3] T. K. Dey and J. Pach. Extremal problems for geometric hypergraphs. *Algorithms and Computation* (Proc. ISAAC '96, Osaka; T. Asano et al., eds.), Lecture Notes in Computer Science 1178, Springer-Verlag, 105-114 (1996). Also in: *Discrete and Computational Geometry* 19, 473-484 (1998).

- [4] R. Fabila-Monroy and J. López. Computational search of small point sets with small rectilinear crossing number. *Journal of Graph Algorithms and Applications*, DOI: 10.7155/jgaa.00328 (2014).
- [5] O. Güler. *Foundations of Optimization*. Springer Science and Business Media, 2010.
- [6] X. He and M. Y. Kao. Regular edge labelings and drawings of planar graphs. *Graph Drawing*, 96-103 (1995).
- [7] P. C. Kainen. The book thickness of a graph II. *Congressus Numerantium*, 71, 121-132 (1990).
- [8] J. Matoušek. *Lectures in Discrete Geometry*. Springer, 2002.
- [9] P. McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17, 179-184 (1970).
- [10] M. Schaefer. The graph crossing number and its variants: a survey. *The Electronic Journal of Combinatorics, Dynamic Survey* 21 (2014).
- [11] F. Shahrokhi and O. Sykora and L. Szekely and I. Vrto. The gap between the crossing numbers and the convex crossing numbers. *Contemporary Mathematics*, 342, 249-258 (2004).
- [12] I. G. Tollis and C. Xia. Drawing telecommunication networks. *Graph Drawing*, 206-217 (1995).